Non-existence of solutions to fractional stochastic heat equations

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The Osgood condition for ODEs

- **Assumption 1 :** $b : \mathbf{R} \to \mathbf{R}_+$ is nonegative, locally Lipschitz and nondecreasing

- Consider the ODE

$$X_t = a + \int_0^t b(X_s)\,\mathrm{d}s, \quad t \geq 0, \quad a \geq 0.$$

  This equation admits a unique solution up to its blow up time

$$T := \sup\{t > 0 : |X_t| < \infty\} = \int_a^\infty \frac{1}{b(s)}\,\mathrm{d}s.$$

  We say that the solution blows up in finite time if $T < \infty$.

The Osgood condition for integral equations

- **The Osgood condition** : for some $a > 0$

$$\int_a^\infty \frac{1}{b(s)}\,\mathrm{d}s < \infty.$$

- **Assumption 2 :** $g : [0, \infty) \to \mathbf{R}$ is continuous and

$$\limsup_{t\to\infty} \inf_{0\le h\le 1} g(t+h) = \infty.$$

Theorem (León-Villa'11)

Suppose that Assumptions 1 and 2 hold. The solution to

$$X_t = a + \int_0^t b(X_s)\,\mathrm{d}s + g(t), \quad a \geq 0,$$

blows up in finite time if and only if the Osgood condition holds.

Proof (recall $X_t = a + \int_0^t b(X_s)\,\mathrm{d}s + g(t)$)

- Assume $T < \infty$. Set $M := \sup_{0 \le s \le T} |g(s)|$. For $t \in [0, T]$,

$$X_t \le a + M + \int_0^t b(X_s)\,\mathrm{d}s.$$

Let

$$Y_t = a + M + 1 + \int_0^t b(Y_s)\,\mathrm{d}s.$$

Then $X_t \le Y_t$ on $[0, T]$.

So Y_t will also blow up by time T and b satisfies the Osgood condition.

Proof (recall $X_t = a + \int_0^t b(X_s)\,\mathrm{d}s + g(t)$)

- Suppose $T = \infty$. Let $t_n \to \infty$. Then, for $t \in [0,1]$,

$$X_{t+t_n} \geq a + \int_{t_n}^{t+t_n} b(X_s)\,\mathrm{d}s + g(t+t_n)$$
$$\geq a + \int_0^t b(X_{s+t_n})\,\mathrm{d}s + \inf_{0\leq h\leq 1} g(h+t_n),$$

This means that $X_{t+t_n} \geq Z_t$ where

$$Z_t = \frac{1}{2}\left(a + \inf_{0\leq h\leq 1} g(h+t_n)\right) + \int_0^t b(Z_s)\,\mathrm{d}s.$$

In particular,

$$\int_{\frac{1}{2}(a+\inf_{0\leq h\leq 1} g(h+t_n))}^{\infty} \frac{1}{b(s)}\,\mathrm{d}s > 1.$$

But from Assumption 2, we can find $t_n \to \infty$ such that

$$\frac{1}{2}(a + \inf_{0\leq h\leq 1} g(h+t_n)) \to \infty.$$

This contradicts the Osgood condition.

Bifractional Brownian motion

- Introduced by Houdré-Villa'03 is defined as a centered Gaussian process $(B_t^{H,K})_{t\geq 0}$ with covariance

$$R^{H,K}(t,s) = 2^{-K}((t^{2H} + s^{2H})^K - |t-s|^{2HK}),$$

where $H \in (0,1)$ and $K \in (0,1]$. $B_t^{H,1}$ is a fBm.

- Set

$$\psi_{H,K}(t) := t^{HK}\sqrt{2\log\log t}, \quad t > e.$$

Lemma (LIL, Arcones'95)

Almost surely,

$$\limsup_{t\to\infty} \frac{B_t^{H,K}}{\psi_{H,K}(t)} = 1.$$

SDEs driven by bifractional Brownian motion

Lemma (León-Villa'11)

Almost surely, there exists $t_n \to \infty$ such that

$$\inf_{h\in[0,1]} B^{H,K}_{t_n+h} \to \infty \quad \text{as} \quad n \to \infty.$$

Theorem (León-Villa'11)

Suppose that Assumptions 1 holds. Then the solution to

$$X_t = a + \int_0^t b(X_s)\,\mathrm{d}s + B^{H,K}_t, \quad a \geq 0,$$

blows up in finite time almost surely if and only if the Osgood condition holds.

Proof (of $\inf_{h\in[0,1]} B^{H,K}_{t_n+h} \to \infty$)

- One first shows that a.s.

$$\sup_{s,t\in[n,n+2]} \frac{|B^{H,K}_t - B^{H,K}_s|}{\psi_{H,K}(n)} \longrightarrow 0, \quad \text{as } n\to\infty. \tag{1}$$

In fact,

$$\mathrm{E}\left[\sum_{n=1}^{\infty} \sup_{s,t\in[n,n+2]} \frac{|B^{H,K}_t - B^{H,K}_s|^p}{\psi_{H,K}(n)^p}\right] \le \sum_{n=1}^{\infty} \frac{A_p 2^{pHK}}{\psi_{H,K}(n)^p} < \infty.$$

- Let ω such that both LIL and (1) hold. Then

$$\begin{aligned}\inf_{h\in[0,1]} B^{H,K}_{t+h} &\ge B^{H,K}_t + \inf_{h\in[0,1]}\left(-|B^{H,K}_{t+h} - B^{H,K}_t|\right)\\ &\ge \frac{B^{H,K}_t}{\psi_{H,K}(t)}\psi_{H,K}(t) - \sup_{h\in[0,1]} \frac{|B^{H,K}_{t+h} - B^{H,K}_t|}{\psi_{H,K}([t])}\psi_{H,K}([t]).\end{aligned}$$

Finally, LIL and (1) conclude the proof.

The stochastic heat equation on $[0, 1]$

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \Delta u(t,x) + b(u(t,x)) + \sigma \dot{W}(t,x), \quad x \in [0,1], \; t > 0, \\ u(0,x) = u_0(x), \end{cases}$$

- homogeneous Dirichlet boundary conditions, $\sigma > 0$, $\dot{W}$ space-time white noise
- $u_0(x)$ nonnegative and continuous function

- If b is locally Lipschitz then there exists a unique local random field solution which is a jointly measurable and adapted space-time process satisfying

$$u(t,x) = \int_0^1 p(t,x,y) u_0(y)\,\mathrm{d}y + \int_0^t \int_0^1 p(t-s,x,y) b(u(s,y))\,\mathrm{d}y\,\mathrm{d}s + \sigma \int_0^t \int_0^1 p(t-s,x,y) W(\mathrm{d}y\,\mathrm{d}s),$$

for all $t \in (0, T)$, where $T = \sup\{t > 0 : \sup_{x \in [0,1]} |u(t,x)| < \infty\}$ and $p(t,x,y)$ is the Dirichlet heat kernel on $[0, 1]$.

Blow up results

$$\frac{\partial u(t,x)}{\partial t} = \Delta u(t,x) + b(u(t,x)) + \sigma \dot{W}(t,x), \quad x \in [0,1],\ t > 0.$$

Theorem (Bonder-Groisman'09)

If b is nonnegative, locally Lipschitz, convex, and satisfies the Osgood condition, then the solution blows up in finite time.

Theorem (Dalang-Khoshnevisan-Zhang'19)

If b is locally Lipschitz and $|b(x)| = O(|x| \log |x|)$ *as* $|x| \to \infty$*, then there exists a global solution.*

Observe that if $b(x) \sim |x|(\log |x|)^{\delta}$, as $x \to \infty$, the Osgood condition holds iff $\delta < 1$.

Thus, Dalang-Khoshnevisan-Zhang'19 result shows that the Osgood condition is optimal.

Our blow up result= converse of Bonder-Groisman

$$\frac{\partial u(t,x)}{\partial t} = \Delta u(t,x) + b(u(t,x)) + \sigma \dot{W}(t,x), \quad x \in [0,1], \; t > 0.$$

Theorem (Foondun-Nualart'20)

Suppose that Assumption 1 holds. If the solution blows up in finite time with positive probability then b satisfies the Osgood condition.

Observe that Bonder-Groisman's Theorem shows that if $b(u) = u^{1+\eta}$, with $\eta > 0$, then there is no global solution no matter how small the initial condition is.

When $\sigma = 0$, for any $\eta > 0$ one can construct nontrivial global solutions by taking u_0 small enough.

Proof (necessity of the Osgood condition)

- Assume $\mathrm{P}(T < \infty) > 0$. We can find Ω satisfying $\mathrm{P}(\Omega) > 0$ such that for any $\omega \in \Omega$, we have $T(\omega) < \infty$.
- Set
$$M := \sup_{x\in[0,\,1]} \sup_{t\in(0,T]} \left| \int_0^t \int_0^1 p(t-s,\, x,\, y) W(\mathrm{d}y\, \mathrm{d}s) \right| .$$
- Set $Y_t := \sup_{x\in[0,\,1]} u(t,\, x)$. Then,
$$Y_t \leq a + \sigma M + K + \int_0^t b(Y_s)\, \mathrm{d}s.$$
Then, as in the proof of León-Villa's theorem we conclude that the Osgood conditions holds.

The fractional stochastic heat equation on $B_1(0)$

$$\left| \begin{array}{l} \dfrac{\partial u(t,x)}{\partial t} = \mathcal{L}u(t,x) + b(u(t,x)) + \dot{F}(t,\,x), \quad x \in B_1(0),\; t > 0, \\ u(0,\,x) = u_0(x), \end{array} \right.$$

- $\mathcal{L}$ is the generator of a symmetric α-stable process killed upon exiting the ball $B_1(0)$, $\alpha \in (0,2]$.
- homogeneous Dirichlet boundary conditions $u(t,x) = 0, x \in \mathbf{R}^d \setminus B_1(0), t > 0$
- $\dot{F}$ is a Gaussian noise which is white in time and has a spatial correlation f whose Fourier transform $\mathcal{F}f = \mu$ is a tempered measure satisfying

$$\int_{\mathbf{R}^d} \frac{\mu(\mathrm{d}\xi)}{(1+|\xi|^\alpha)^\rho} < \infty,$$

for some $\rho \in (0,1)$ (Sanz-Solé-Sarrà'02).

The fractional stochastic heat equation on $B_1(0)$

- As before, if b is locally Lipschitz then there exists a unique local random field solution satisfying

$$u(t,x) = \int_{B_1(0)} p_\alpha(t,x,y)u_0(y)\,\mathrm{d}y + \int_0^t \int_{B_1(0)} p_\alpha(t-s,x,y)b(u(s,y))\,\mathrm{d}y\,\mathrm{d}s + \int_0^t \int_{B_1(0)} p_\alpha(t-s,x,y)F(\mathrm{d}y\,\mathrm{d}s),$$

for $t \in (0,T)$, where $T = \sup\{t > 0 : \sup_{x \in B_1(0)} |u(t,x)| < \infty\}$ and $p_\alpha(t,x,y)$ is the Dirichlet fractional heat kernel on $B_1(0)$.

Theorem (Foondun-Nualart'20)

Suppose that Assumption 1 holds. Then if b satisfies the Osgood condition and is convex, the solution blows up in finite time almost surely. On the other hand, if the solution blows up in finite time with positive probability, then b satisfies the Osgood condition.

Proof (of the sufficiency of the Osgood condition)

- Recall that

$$p_\alpha(t, x, y) = \sum_{n=1}^{\infty} e^{-\lambda_n t} \phi_n(x)\phi_n(y)$$

where $\{\phi_n\}_{n\geq 1}$ is an orthonormal basis of $L^2(B_1(0))$ and $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$

$$\begin{cases} -(-\Delta)^{\alpha/2}\phi_n(x) = -\lambda_n\phi_n(x) & x \in B_1(0), \\ \phi_n(x) = 0 & x \in \mathbf{R}^d \setminus B_1(0). \end{cases}$$

- Set

$$Y_t = c\int_{B_1(0)} u(t,x)\phi_1(x)\mathrm{d}x$$

where $c^{-1} = \int_{B_1(0)} \phi_1(x)dx$.

- Then

$$Y_t = e^{-\lambda_1 t}(Y_0 + \int_0^t e^{\lambda_1 s}\int_{B_1(0)} \phi_1(y)b(u(s,y))\mathrm{d}y\mathrm{d}s + \int_0^t e^{\lambda_1 s}\int_{B_1(0)} \phi_1(y)F(\mathrm{d}y\,\mathrm{d}s)).$$

Proof (of the sufficiency of the Osgood condition)

- Recall

$$Y_t = e^{-\lambda_1 t}\Big(Y_0 + \int_0^t e^{\lambda_1 s}\int_{B_1(0)} \phi_1(y) b(u(s,y))\mathrm{d}y\mathrm{d}s + \int_0^t e^{\lambda_1 s}\int_{B_1(0)} \phi_1(y) F(\mathrm{d}y\,\mathrm{d}s)\Big).$$

- Since b is convex, by Jensen's inequality,

$$\int_{B_1(0)} \phi_1(y) b(u(s,y))\mathrm{d}y \geq b(Y_s).$$

- Then we obtain that $Y_t \geq X_t$ a.s., where

$$dX_t = (-\lambda_1 X_t + b(X_t))\mathrm{d}t + \mathrm{d}Z_t, \qquad X_0 = Y_0$$

and

$$Z_t := c\int_0^t \int_{B_1(0)} c\phi_1(y) F(\mathrm{d}y\,\mathrm{d}s).$$

- Finally, we use Feller's test for explosion as $Z_t = \sqrt{\kappa} B_t$, where B_t is a Brownian motion and

$$\kappa := c^2 \int_{B_1(0)\times B_1(0)} \phi_1(y)\phi_1(z) f(y-z)\mathrm{d}y\mathrm{d}z.$$

Extension to multiplicative noise

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \mathcal{L}u(t,x) + b(u(t,x)) + \sigma(u(t,x))\dot{F}(t,x), & x \in B_1(0),\ t > 0, \\ u(0,x) = u_0(x), \end{cases}$$

- σ is a locally Lipschitz function satisfying $\frac{1}{K} \leq \sigma(x) \leq K$ for all $x \in \mathbf{R}$ and some $K > 0$.

Theorem (Foondun-Nualart'20)

Suppose that Assumption 1 holds. If the solution blows up in finite time with positive probability, then b satisfies the Osgood condition.

The stochastic heat equation on $\mathbf{R}$

$$\left|\begin{aligned} \frac{\partial u(t,x)}{\partial t} &= \Delta u(t,x) + b(u(t,x)) + \sigma \dot{W}(t,x) \quad x \in \mathbf{R},\ t>0,\\ u(0,x) &= u_0(x). \end{aligned}\right.$$

Theorem (Foondun-Nualart'20)

Suppose that Assumption 1 holds. Then, if b satisfies the Osgood condition, then almost surely, there is no global solution.

This Theorem shows that if $b(u) = u^{1+\eta}$, with $\eta > 0$, then there is no global solution meaning that there is no Fujita exponent in the stochastic setting.

Recall that when $\sigma = 0$ and $x \in \mathbf{R}^d$, if $\eta > 2/d$, one can construct nontrivial global solutions when u_0 is small enough (Fujita'66).

The stochastic heat equation on $\mathbf{R}$

- The mild formulation writes as

$$u(t, x) = \int_{\mathbf{R}} G(t, x, y) u_0(y)\, \mathrm{d}y + \int_0^t \int_{\mathbf{R}} G(t-s, x, y) b(u(s, y))\, \mathrm{d}y\, \mathrm{d}s + \sigma g(t, x)$$

where $G(t, x, y)$ is the heat kernel and

$$g(t, x) := \int_0^t \int_{\mathbf{R}} G(t-s, x, y) W(\mathrm{d}y\, \mathrm{d}s).$$

- For a fixed $x \in \mathbf{R}$, the process $(g(t, x), t \geq 0)$ is a bifractional Brownian motion with parameters $H = K = \frac{1}{2}$ multiplied by a constant (Lei-D.Nualart'09).

Theorem (Foondun-Nualart'20)

A.s. there exists $t_n \to \infty$ such that

$$\inf_{h \in [0,1], x \in [0,1]} g(t_n + h, x) \to \infty \quad \text{as} \quad n \to \infty.$$

Proof (of $\inf_{h\in[0,1],x\in[0,1]} g(t_n+h,x)\to\infty$)

- Using an improvement of the classical Garsia's lemma obtained in Dalang-khosnevisan-Nualart'07 we show that for all $p\geq 2$ and integer $n\geq 1$,

$$\mathrm{E}\left[\sup_{s,t\in[n,n+2],x,y\in[0,1]}|g(t,x)-g(s,y)|^p\right]\leq A_p 2^{p/4}.$$

- As a consequence, a.s.

$$\sup_{s,t\in[n,n+2],x,y\in[0,1]}\frac{|g(t,x)-g(s,y)|}{\psi_{\frac{1}{2},\frac{1}{2}}(n)}\longrightarrow 0, \quad \text{as } n\to\infty.$$

- Fix $x_0\in[0,1]$ and write

$$\begin{aligned}\inf_{h\in[0,1],x\in[0,1]} g(t+h,x) &\geq g(t,x_0)+\inf_{h\in[0,1],x\in[0,1]}\left(-|g(t+h,x)-g(t,x_0)|\right)\\ &\geq \frac{g(t,x_0)}{\psi_{\frac{1}{2},\frac{1}{2}}(t)}\psi_{\frac{1}{2},\frac{1}{2}}(t)-\sup_{h\in[0,1],x\in[0,1]}\frac{|g(t+h,x)-g(t,x_0)|}{\psi_{\frac{1}{2},\frac{1}{2}}([t])}\psi_{\frac{1}{2},\frac{1}{2}}([t]).\end{aligned}$$

- Using the LIL for bifBm, we conclude the proof.

Proof (of the sufficiency of the Osgood condition)

- Assume that there is a global solution a.s. Let $t_n \to \infty$. Then

$$u(t+t_n,\, x) \geq \int_{\mathbf{R}} G(t+t_n,\, x,\, y) u_0(y)\,\mathrm{d}y + \sigma g(t+t_n, x) + \int_0^t \int_{\mathbf{R}} G(t-s,\, x,\, y) b(u(s+t_n,\, y))\,\mathrm{d}y\,\mathrm{d}s.$$

- There exists $t_n \to \infty$ such that $g(t+t_n, x) > 0$ for all $x \in (0,\,1)$ and $t \in [0,1]$, and thus $u(t+t_n,\, x) > 0$ as well.

- For fixed $x \in (0,\,1)$ and $t \in [0,1]$,

$$\begin{aligned}\int_0^t \int_{\mathbf{R}} G(t-s,\, x,\, y) b(u(s+t_n,\, y))\,\mathrm{d}y\,\mathrm{d}s \\ &\geq \int_0^t b\left(\inf_{y\in(0,\,1)} u(s+t_n,\, y)\right) \int_{(0,\,1)} G(t-s,\, x,\, y)\,\mathrm{d}y\,\mathrm{d}s \\ &\geq \int_0^t b\left(\inf_{y\in(0,\,1)} u(s+t_n,\, y)\right)\,\mathrm{d}s,\end{aligned}$$

as $G(t,\, x,\, y) \geq \frac{c}{t^{1/2}}$ whenever $|x-y| \leq t^{1/2}$.

Proof (of the sufficiency of the Osgood condition)

- Set $Y_t := \inf_{y \in (0,1)} u(t + t_n, y)$.
- We have shown that

$$Y_t \geq \inf_{0 \leq h \leq 1, x \in (0,1)} \left\{ \int_{\mathbf{R}} G(h + t_n, x, y) u_0(y) \, \mathrm{d}y + \sigma g(h + t_n, x) \right\} + \int_0^t b(Y_s) \, \mathrm{d}s.$$

- Using the last Theorem, we conclude that the Osgood condition cannot hold.

The fractional stochastic heat equation on $\mathbf{R}^d$

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \mathcal{L}u(t,x) + b(u(t,x)) + \sigma \dot{F}(t,\, x), \quad x \in \mathbf{R}^d,\ t > 0, \\ u(0,\, x) = u_0(x), \end{cases}$$

- $\mathcal{L}$ is the generator of a symmetric α-stable process, $\alpha \in (0,2]$.
- $\dot{F}$ is a Gaussian noise which is white in time and has a spatial correlation f given by the Riesz kernel $f(x) = |x|^{-\beta}$, $0 < \beta < \alpha$.

Theorem (Foondun-Nualart'20)

Suppose that Assumption 1 holds. Then, if b satisfies the Osgood condition, then almost surely, there is no global solution.

The fractional stochastic heat equation on $\mathbf{R}^d$

- The mild formulation writes as

$$u(t,x) = \int_{\mathbf{R}^d} G_\alpha(t,x,y)u_0(y)\mathrm{d}y + \int_0^t \int_{\mathbf{R}^d} G_\alpha(t-s,x,y)b(u(s,y))\mathrm{d}y\mathrm{d}s + \sigma g_{\alpha,\beta}(t,x)$$

  where $G_\alpha(t,\ x,\ y)$ is the fractional heat kernel and

$$g_{\alpha,\beta}(t,x) := \int_0^t \int_{\mathbf{R}^d} G_\alpha(t-s,\ x,\ y)F(\mathrm{d}y\,\mathrm{d}s).$$

- For a fixed $x \in \mathbf{R}^d$, the process $(g_\alpha(t,x))_{t\geq 0}$ is a bifractional Brownian motion with parameters $H = \frac{\alpha-\beta}{2}$ and $K = \frac{1}{\alpha}$, multiplied by a constant.

Theorem (Foondun-Nualart'20)

A.s. there exists $t_n \to \infty$ such that

$$\inf_{h\in[0,1],x\in B_1(0)} g_{\alpha,\beta}(t_n+h,x) \to \infty \quad \textit{as} \quad n \to \infty.$$

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